Parabolic Subgroups

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The origin of the name parabolic is a mystery. Borel in his history [Bor01, VI.§2] attributes it to R. Godement in [God61]. Godement conjectures that the quotient $G(\mathbb{A})/G(\mathbb{Q})$ is compact if and only if every element of $G(\mathbb{Q})$ is semi-simple, as is the case in classical groups (this was shortly thereafter proven [MT62]). He says that

> "Lorsque n'est pas compact, il est non moins facile de conjecturer qu'on doit pouvoir définir quelque chose d'analogue aux classiques "pointes paraboliques", lesquelles doivent correspondre à des sous-groupes unipotents non triviaux de $G_{\mathbb{Q}}$"

which roughly (google) translates to that one can also conjecture that non-trivial unipotent elements should correspond to "parabolic points" in a fundamental domain.

In the case of modular forms the fundamental domain is $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. We have the classification of elements of $\mathrm{SL}_2(\mathbb{R})\backslash\{\pm 1\}$ as in [Bor97, 3.5] via their trace

$$g \text{ is of type } \quad \begin{cases} \text{Elliptic if} & \frac{1}{2}|tr(g)| < 1 \\ \text{Parabolic if} & \frac{1}{2}|tr(g)| = 1 \\ \text{Hyperbolic if} & \frac{1}{2}|tr(g)| > 1 \end{cases}.$$

This classification, it seems, relies entirely on the *aesthetic* connection with the classification of the sections of conics via eccentricity. Proper parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$ can be realised as the stabilisers of lines in $\mathbb{R}^2$ under the standard action of SL_2 on $\mathbb{R}^2$ [Bor97, 2.6] and moreover an element of $\mathrm{SL}_2(\mathbb{R})$ is parabolic if and only if it has one fixed point on $\partial\bar{\mathcal{H}}$ and none on $\mathcal{H}$ [Bor97, 3.5].

Being parabolic is equivalent to having eigenvalue 1 hence by the Jordan decomposition we know that parabolics in SL_2 are conjugate (over $\mathbb{C}$) to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly the standard parabolic

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R}),$$

contains these matrices, and moreover all parabolics are *conjugate* to this parabolic. Hence all parabolic elements are contained in a parabolic subgroup.

The take away is that perhaps the folklore of the name being for "para-Borelic", as in kind of a Borel, is probably a better way of thinking of them.

References

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[MT62] G. D. Mostow and T. Tamagawa. On the Compactness of Arithmetically Defined Homogeneous Spaces. *Annals of Mathematics*, 76(3):446–463, 1962.